AIAA 81-0523R

Vibration of Some Structures with Periodic Random Parameters

Wen-hu Huang*
Harbin Institute of Technology, Harbin, China

In a periodic structural system, such as blades in a closed packet of turbomachinery, the natural frequencies of the individual blades can be randomly different from one another. This paper describes a solution for such a periodic structure in which the distributions of blade frequencies are random processes with small standard deviations. A spectral method is suggested to solve differential equations with random coefficients. The expressions for vibration modes are given; the standard deviations of natural frequencies are estimated, and the results of forced vibration are presented. Some special features of vibration characteristics of this system are shown as well.

Nomenclature

a,b,c,d,A,B	= amplitude coefficients
f	= Fourier coefficient of randomly distributed stiffness
g	= Fourier coefficient of randomly distributed stiffness of connecting elements
h	= Fourier coefficient of randomly distributed mass
$c_x(\theta),k_x(\theta)$	= distributed stiffnesses
$d_x(\theta)$	= distributed damping
$f(\theta,t)$	= exciting-force function
F	= amplitude of exciting force
k	= number of harmonics of exciting force
ℓ	= number of Fourier coefficients
m,n,r	= number of nodal diameters of normal modes
$m_x(\theta)$	= function of distributed mass
m_{x0}	= mean value of mass
$M(\theta)$	= random function of $m_x(\theta)$
$p(\theta),q(\theta)$	= local frequencies
p_0,q_0	= mean values of $p(\theta)$ and $q(\theta)$, respectively
$P(\theta),Q(\theta)$	= random functions of $p(\theta)$ and $q(\theta)$, respectively
$R(\tau)$	= correlation function
t	= time
$u(\theta),v(\theta)$	= components of mode due to random stiffness
$w(\theta)$	= component of mode due to random mass
$x(\theta,t)$	= displacement function
$X(\theta)$	= normal mode of the system
$y(\theta)$	= normal mode of the structure with homogeneous parameters
$\alpha,\beta,\gamma,\delta$	= phase angles
δ_{mn}	$= \begin{cases} 1 \text{ (for } m=n) \\ 0 \text{ (for } m\neq n) \end{cases}$
ϵ	= damping factor
θ,τ	= spatial angular coordinates
$\sigma_m,\sigma_p,\sigma_q$	= standard deviations of $M(\theta)$, $P(\theta)$, and $Q(\theta)$, respectively
ξ,η,ζ	= small parameters
ω	= natural frequency of the structure
λ,μ,ν,ρ	= natural frequencies corresponding to $y(\theta)$, $u(\theta)$, $v(\theta)$, and $w(\theta)$, respectively
Ω	= frequency of exciting force

Presented as Paper 81-0523 at the AIAA/ASME/ASCE/AHS 22nd Structures, Structural Dynamics and Materials Conference, Atlanta, Ga., April 6-8, 1981; submitted April 21, 1981; revision received Oct. 23, 1981.

*Professor, Dept. of Applied Mechanics.

Introduction

IN stochastic structural dynamics, the most widely studied problem is the vibration of structures with deterministic parameters under the action of randomly time-varying exciting forces. The vibration problems of dynamic system with random structural parameters are not yet fully analyzed. Hoshiya and Shah[1] solved the problem of free vibration of a beam column with random structural parameters. In their paper the auto- and cross-correlation functions of the random structural parameters are artificially assumed to be of the form of exponential functions.

This paper deals with the free and forced vibration of some structures with periodic random parameters. An example of such a structure is the blades on the disk of a turbomachine connected to one another by elements to form a circumferentially closed structure, which is described in Ref. 2. The natural frequencies of the individual blades can differ from one another randomly. This structure may be modeled by a stiff ring supported by transverse springs with randomly distributed stiffness and mass parameters. First we assume that the standard deviations of these random parameters are small, so this problem may be solved asymptotically by the perturbation method. Then, considering that the structure is circumferentially closed and that the random functions of structural parameters are periodic, it is reasonable to introduce a spectral method of expanding these periodic random functions into Fourier series. Thus, in this problem the solutions of differential equations may be expressed by Fourier coefficients to obtain the expressions of natural frequencies, normal modes, and amplitudes in resonance and to estimate their variances.

Governing Equations

Consider a circumferentially closed packet of turbomachinery blades on a disk.[2] When the packet vibrates in one of its normal modes, every blade may be modeled as a weightless cantilever bar with a concentrated mass at its free end. These masses are connected by springs to form a closed ring (Fig. 1). The motion of this model of blades can be described by the following equation:

$$m_{x_j}\frac{\mathrm{d}^2x_j}{\mathrm{d}t^2}+k_{x_j}x_j-c_{x_j}(x_{j+1}-x_j)$$
$$+c_{x_{j-1}}(x_j-x_{j-1})+d_{x_j}\frac{\mathrm{d}x_j}{\mathrm{d}t}=f_{x_j}(t) \tag{1}$$

where $x_j(t)$ is the displacement of blade j; m_{x_j}, k_{x_j}, c_{x_j}, d_{x_j} the mass, spring stiffness, stiffness of connecting elements, and damping, respectively, of blade j; $f_{x_j}(t)$ the exciting force on blade j.

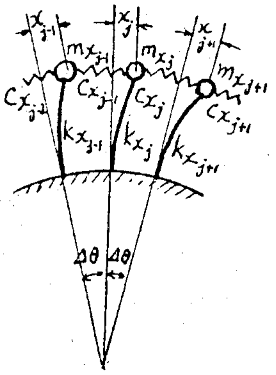

Fig. 1 Closed structure of blades.

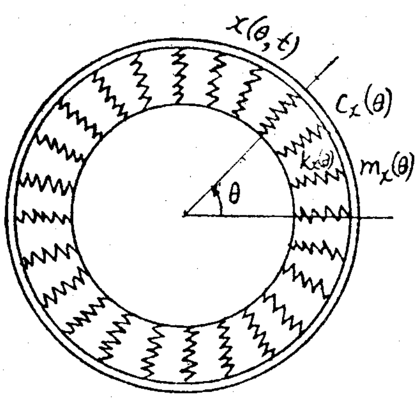

Fig. 2 Closed-ring model.

If we assume that the distribution of blades on the disk is very dense, the above structure may be modeled by a closed ring composed of stiffened string supported by transverse springs (Fig. 2), and Eq. (1) of finite differences may be approximately replaced by the following governing differential equation:

$$m_x \frac{\partial^2 x}{\partial t^2} + k_x x - \frac{\partial}{\partial \theta} c_x \frac{\partial x}{\partial \theta} + d_x \frac{\partial x}{\partial t} = f_x(\theta, t) \tag{2}$$

with continuity conditions

$$x(-\pi, t) = x(\pi, t); \qquad \frac{\partial x}{\partial \theta}\bigg|_{(-\pi,t)} = \frac{\partial x}{\partial \theta}\bigg|_{(\pi,t)}$$

and initial conditions

$$x(\theta, 0) = x_0(\theta); \qquad \frac{\partial x}{\partial t}\bigg|_{(\theta,0)} = \dot{x}_0(\theta)$$

where $m_x(\theta)$, $k_x(\theta)$, $c_x(\theta)$, and $d_x(\theta)$ are functions of distributed mass, spring stiffness, string stiffness, and damping, respectively.

Assume that the system carries on a harmonic vibration under the action of a harmonic exciting force. The force can be expressed by

$$f_x(\theta, t) = m_{x_0} F(\theta) e^{i\omega t} \tag{3}$$

and the steady solution of differential Eq. (2) is

$$x(\theta, t) = X(\theta) e^{i\omega t} \tag{4}$$

Introduce the following notations:

$$q^2(\theta) = \frac{1}{m_{x_0}} c_x(\theta); \qquad p^2(\theta) = \frac{1}{m_{x_0}} k_x(\theta); \qquad \epsilon = \frac{d_x}{m_{x_0}}$$

These are the squares of local frequencies and the damping factor, respectively. Thus we get the spatial differential equation

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\left(q^2 \frac{\mathrm{d}X}{\mathrm{d}\theta}\right) + \left(p^2 - \frac{m_x}{m_{x_0}}\omega^2\right)X + i\epsilon\omega X = F(\theta) \tag{5}$$

Random Structural Parameters and Their Correlation Functions

The structural parameters $p^2(\theta)$, $q^2(\theta)$, and $m_x(\theta)$ are periodic random functions fluctuating around their respective mean values p_0^2, q_0^2, and m_{x0}. Here, to simplify the discussion we assume the damping factor ϵ to be constant. Assume that the fluctuations of these structural parameters are small. Introducing small parameters ξ, η, and ζ, these structural parameters may be expressed in the following form:

$$p^2(\theta) = p_0^2[1 + \xi P(\theta)]$$

$$q^2(\theta) = q_0^2[1 + \eta Q(\theta)]$$

$$m_x(\theta) = m_{x_0}[1 + \zeta M(\theta)] \tag{6}$$

where $P(\theta)$, $Q(\theta)$, and $M(\theta)$ are periodic stationary random processes and their variances are assumed equal to 1. They satisfy the following conditions of periodicity:

$$P(-\pi) = P(\pi); \qquad Q(-\pi) = Q(\pi)$$

$$Q'(-\pi) = Q'(\pi); \qquad M(-\pi) = M(\pi) \tag{6$'$}$$

Later on we shall make use of the periodicity of these functions to establish a spectral method. We expand these periodic functions into Fourier series:

$$P(\theta) = \sum_{\ell=-\infty}^{\infty} f_\ell e^{i\ell\theta} \qquad Q(\theta) = \sum_{\ell=-\infty}^{\infty} g_\ell e^{i\ell\theta} \qquad M(\theta) = \sum_{\ell=-\infty}^{\infty} h_\ell e^{i\ell\theta} \tag{7}$$

where f_ℓ, g_ℓ, and h_ℓ are random Fourier coefficients and the $(-\ell)$th coefficient of any of them equals the conjugate of their ℓth coefficient, i.e.,

$$f_{-\ell} = f_\ell^*; \qquad g_{-\ell} = g_\ell^*; \qquad h_{-\ell} = h_\ell^* \tag{7$'$}$$

Assume random processes $P(\theta)$, $Q(\theta)$, and $M(\theta)$ to be ergodic. The autocorrelation function $R_p(\tau)$ of $P(\theta)$ can be obtained as follows:

$$R_p(\tau) = E[P(\theta)P(\theta+\tau)]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{\ell=-\infty}^{\infty} f_\ell e^{i\ell\theta} \sum_{\ell'=-\infty}^{\infty} f_{\ell'} e^{i\ell'(\theta+\tau)} \mathrm{d}\theta$$

$$= \sum_{\ell=0}^{\infty} f_\ell f_\ell^*(e^{i\ell\tau} + e^{-i\ell\tau}) = 2\sum_{\ell=0}^{\infty} |f_\ell|^2 \cos\ell\tau \tag{8}$$

When $\tau = 0$, the value of the correlation function is equal to the variance of $P(\theta)$, which is equal to 1 as mentioned above:

$$R_p(0) = 2\sum_{\ell=0}^{\infty} |f_\ell|^2 = 1 \tag{8$'$}$$

We can get similar results for autocorrelation functions $R_q(\tau)$ and $R_m(\tau)$ for functions $Q(\theta)$ and $M(\theta)$, respectively. Assume that the variances of random processes $\xi P(\theta)$, $\eta Q(\theta)$, and $\zeta M(\theta)$ are σ_p^2, σ_q^2, and σ_m^2, respectively. Then from the above analysis we see that the small parameters ξ, η, and ζ are equal to the standard deviations, respectively:

$$\xi=\sigma_p; \quad \eta=\sigma_q; \quad \zeta=\sigma_m \tag{9}$$

Now let us compare the autocorrelation function (8) with that of Hoshiya and Shah,[1] who suggested the following form:

$$R_p(\tau)=e^{-\alpha|\tau|} \tag{10}$$

As we know, the Fourier transform of the autocorrelation function is equal to the power spectral density. Compare the Fourier transforms of Eqs. (8) and (10); we see that

$$\begin{aligned}|f_\ell|^2 &= \frac{\alpha}{\pi(\alpha^2+\ell^2)}(1+e^{-\alpha\pi}) \quad \text{for} \quad \ell=1,3,5,\ldots \\ &= \frac{\alpha}{\pi(\alpha^2+\ell^2)}(1-e^{-\alpha\pi}) \quad \text{for} \quad \ell=2,4,\ldots \\ &= \frac{2}{\pi\alpha}(1-e^{-\alpha\pi}) \qquad \text{for} \quad \ell=0\end{aligned} \tag{11}$$

As an example, we put $\alpha=1$; then the Fourier coefficient f takes the following values (which are shown in Fig. 3).

Free Vibration

The differential equation of free vibration is written by dropping the exciting fource and neglecting the damping of Eq. (5):

$$-q_0^2[(1+\eta Q)X']'+[p_0^2(1+\xi P)-\omega^2(1+\zeta M)]X=0 \tag{12}$$

with boundary conditions

$$X(-\pi)=X(\pi); \quad X'(-\pi)=X'(\pi) \tag{13}$$

The differential Eq. (12) is solved by the perturbation method, in which the frequency ω^2 and the mode shape function $X(\theta)$ are expressed by a power series of small parameters ξ, η, and ζ. Here we consider an approximate solution and keep only the first-order terms:

$$\omega^2=\lambda^2+\xi\mu^2+\eta\nu^2+\zeta\rho^2+\ldots \tag{14}$$

$$X(\theta)=y(\theta)+\xi u(\theta)+\eta v(\theta)+\zeta w(\theta)+\ldots \tag{15}$$

Thus we get the following differential equations:

$$-q_0^2y''+(p_0^2-\lambda^2)y=0 \tag{16a}$$

$$-q_0^2u''+(p_0^2-\lambda^2)u=-(p_0^2P-\mu^2)y \tag{16b}$$

$$-q_0^2v''+(p_0^2-\lambda^2)v=q_0^2(Qy''+Q'y')+\nu^2y \tag{16c}$$

$$-q_0^2w''+(p_0^2-\lambda^2)w=(\lambda^2M+\rho^2)y \tag{16d}$$

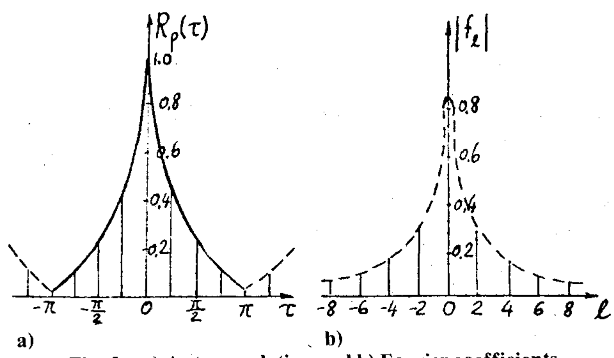

Fig. 3 a) Autocorrelation and b) Fourier coefficients.

Equation (16a) is a differential equation with deterministic coefficients which is easy to solve. The component of frequency of mth order is

$$\lambda_m^2=p_0^2+m^2q_0^2 \tag{17}$$

where m is an integer; $m=0,1,2,\ldots$ equals the number of nodal diameters of the mode. The component of mode function is

$$y_m(\theta)=A_me^{im\theta}+A_m^*e^{-im\theta} \tag{18}$$

Here A_m and A_m^* are conjugate coefficients. By normalization of the orthogonal mode functions

$$\int_{-\pi}^{\pi} y_m(\theta)y_n(\theta)\mathrm{d}\theta=\begin{cases}1 & (\text{for } m=n)\\ 0 & (\text{for } m\neq n)\end{cases} \tag{19}$$

we get

$$A_mA_m^*=|A_m|^2=\frac{1}{4\pi} \tag{20}$$

Once the solution (18) has been obtained, Eqs. (16b-d) can be solved as follows. Take Eq. (16b) as an example. Taking Eq. (17) into account, Eq. (16b) may be rewritten as

$$u_m''+m^2u_m=-\frac{p_0^2}{q_0^2}Py_m-\frac{\mu_m^2}{q_0^2}y_m \tag{21}$$

To solve the above equation, we expand the function $u_m(\theta)$ into a series of determined function $y_r(\theta)$ from Eq. (18):

$$u_m(\theta)=\sum_{r=1}^{\infty}b_{mr}y_r(\theta) \tag{22}$$

where b_{mr} is the coefficient to be determined. Substitute it into Eq. (21), multiply with $y_n(\theta)$, and integrate it with the condition of orthogonality (19); we then get

$$q_0^2(m^2-n^2)b_{mn}+\mu_m\delta_{mn}=p_0^2\int_{-\pi}^{\pi}Py_my_n\mathrm{d}\theta \tag{23}$$

Considering the cases $m=n$ and $m\neq n$, respectively, we get the expressions of component frequency μ_m^2 and mode shape function $u_m(\theta)$ in the following form:

$$\mu_m^2=p_0^2\int_{-\pi}^{\pi}Py_m^2\mathrm{d}\theta=2\pi p_0^2(f_{2m}A_m^{*2}+f_{2m}^*A_m^2) \tag{24}$$

$$u_m(\theta)=\sum_{\substack{n=1\\ n\neq m}}^{\infty}b_{mn}y_n(\theta) \tag{25}$$

where

$$b_{mn}=-b_{nm}=\frac{1}{m^2-n^2}\frac{p_0^2}{q_0^2}\int_{-\pi}^{\pi}Py_my_n\mathrm{d}\theta \tag{26}$$

Assume that

$$A_m=|A_m|e^{i\alpha_m}; \quad f_{2m}=|f_{2m}|e^{i\beta_{2m}}$$

where α_m, β_{2m} are phase angles of these coefficients; thus

$$\mu_m^2=p_0^2|f_{2m}|\cos(\beta_{2m}-2\alpha_m) \tag{27}$$

Solving Eqs. (16c) and (16d) in a similar manner and assuming

$$g_{2m}=|g_{2m}|e^{i\gamma_{2m}}; \quad h_{2m}=|h_{2m}|e^{i\delta_{2m}}$$

we get the component frequencies ν_m^2, ρ_m^2 and mode shape functions $v_m(\theta)$, $w_m(\theta)$ as follows:

$$\nu_m^2 = q_0^2 \int_{-\pi}^{\pi} Qy_m'^2 \mathrm{d}\theta = -m^2 q_0^2 |g_{2m}| \cos(\gamma_{2m} - 2\alpha_m) \qquad \rho_m^2 = -\lambda_m^2 \int_{-\pi}^{\pi} My_m^2 \mathrm{d}\theta = -\lambda_m^2 |h_{2m}| \cos(\delta_{2m} - 2\alpha_m) \tag{28}$$

and

$$v_m(\theta) = \sum_{\substack{n=1 \\ n \neq m}}^{\infty} c_{mn} y_n(\theta) \qquad w_m(\theta) = \sum_{\substack{n=1 \\ n \neq m}}^{\infty} d_{mn} y_n(\theta) \tag{29}$$

where

$$c_{mn} = -c_{nm} = \frac{1}{m^2 - n^2} \int_{-\pi}^{\pi} Qy_m' y_n' \mathrm{d}\theta \qquad d_{mn} = -d_{nm} = -\frac{1}{m^2 - n^2} \frac{\lambda_m^2}{q_0^2} \int_{-\pi}^{\pi} My_m y_n \mathrm{d}\theta \tag{30}$$

Natural Frequency and Its Variance

The mth natural frequency of the system is obtained by synthesizing Eqs. (14), (24), (27), and (28) as follows:

$$\omega_m^2 = (p_0^2 + m^2 q_0^2) + \xi p_0^2 \int_{-\pi}^{\pi} Py_m^2 \mathrm{d}\theta + \eta q_0^2 \int_{-\pi}^{\pi} Qy_m'^2 \mathrm{d}\theta - \zeta\lambda_m^2 \int_{-\pi}^{\pi} My_m^2 \mathrm{d}\theta \tag{31}$$

or

$$\omega_m^2 = (p_0^2 + m^2 q_0^2) + \xi p_0^2 |f_{2m}| \cos(\beta_{2m} - 2\alpha_m) - \eta q_0^2 m^2 |g_{2m}| \cos(\gamma_{2m} - 2\alpha_m) - \zeta\lambda_m^2 |h_{2m}| \cos(\delta_{2m} - 2\alpha_m) \tag{32}$$

The natural frequency ω_m^2 is a random variable. Since the mean values of functions $P(\theta)$, $Q(\theta)$, and $M(\theta)$ equal zero:

$$E[P(\theta)] = 0; \quad E[Q(\theta)] = 0; \quad E[M(\theta)] = 0$$

thus the mean value of ω_m^2 is

$$E[\omega_m^2] = E[\lambda_m^2] + \xi p_0^2 \int_{-\pi}^{\pi} E[P(\theta)] y_m^2 \mathrm{d}\theta + \eta q_0^2 \int_{-\pi}^{\pi} E[Q(\theta)] y_m'^2 \mathrm{d}\theta - \zeta\lambda_m^2 \int_{-\pi}^{\pi} E[M(\theta)] y_m^2 \mathrm{d}\theta = \lambda_m^2 = p_0^2 + m^2 q_0^2 \tag{33}$$

That is to say, the mean value of the natural frequency of the structure with random parameters is equal to the natural frequency of the structure with homogeneous parameters.

As to the variance of natural frequency, it may be estimated by the following integrations from Eq. (31):

$$\begin{aligned} E[(\omega_m^2)^2] = {} & E[(\lambda_m^2)^2] + \xi p_0^4 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[P(\theta_1)P(\theta_2)] y_m^2(\theta_1) y_m^2(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 + \eta q_0^4 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[Q(\theta_1)Q(\theta_2)] y_m'^2(\theta_1) y_m'^2(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \\ & + \zeta\lambda_m^4 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[M(\theta_1)M(\theta_2)] y_m^2(\theta_1) y_m^2(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 + 2\xi\eta p_0^2 q_0^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[P(\theta_1)Q(\theta_2)] y_m^2(\theta_1) y_m'^2(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \\ & - 2\xi\zeta p_0^2 \lambda_m^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[P(\theta_1)M(\theta_2)] y_m^2(\theta_1) y_m^2(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 - 2\eta\zeta q_0^2 \lambda_m^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[Q(\theta_1)M(\theta_2)] y_m'^2(\theta_1) y_m^2(\theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2 \\ & + 2\xi p_0^2 \lambda_m^2 \int_{-\pi}^{\pi} E[P(\theta)] y_m^2(\theta) \mathrm{d}\theta + 2\eta q_0^2 \lambda_m^2 \int_{-\pi}^{\pi} E[Q(\theta)] y_m'^2(\theta) \mathrm{d}\theta - 2\zeta\lambda_m^4 \int_{-\pi}^{\pi} E[M(\theta)] y_m^2(\theta) \mathrm{d}\theta \end{aligned} \tag{34}$$

The variance of ω_m^2 is

$$\begin{aligned} \sigma_\omega^2 = \mathrm{var}(\omega_m^2) = E[(\omega_m^2)^2] - \lambda_m^4 = {} & \tfrac{1}{2}\{\xi^2 p_0^4 |f_{2m}|^2 + \eta^2 m^4 q_0^2 |g_{2m}|^2 + \zeta^2 \lambda_m^4 |h_{2m}|^2 - 2\xi\eta m^2 p_0^2 q_0^2 |f_{2m}| |g_{2m}| \cos(\beta_{2m} - \gamma_{2m}) \\ & - 2\xi\zeta p_0^2 \lambda_m^2 |f_{2m}| |h_{2m}| \cos(\beta_{2m} - \delta_{2m}) + 2\eta\zeta m^2 q_0^2 \lambda_m^2 |g_{2m}| |h_{2m}| \cos(\gamma_{2m} - \delta_{2m})\} \end{aligned} \tag{35}$$

This complicated expression of the standard deviation σ_{ω^2} of ω_m^2 may be considered as a vector sum of the following three vectors (see Fig. 4):

$$\xi p_0^2 f_{2m}, \quad -\eta m^2 q_0^2 g_{2m}, \quad -\zeta\lambda_m^2 h_{2m} \tag{36}$$

Thus the standard deviation of the natural frequency is a small quantity with the same order of ξ, η, and ζ, which are equal to the standard deviations σ_p, σ_q, and σ_m of the structural parameters.

Normal Modes and Their Orthogonality

The normal mode shape function of the system can be obtained by the use of Eqs. (15), (25), and (29). Thus we get

$$X_m(\theta) = y_m(\theta) + \sum_{\substack{n=1\\ n\neq m}}^{\infty} (\xi b_{mn} + \eta c_{mn} + \zeta d_{mn}) y_n(\theta) \tag{37}$$

In Eq. (37) we use

$$y_m(\theta) = A_m e^{im\theta} + A_m^* e^{-im\theta} = 1/(2\sqrt{\pi})\cos(m\theta + \alpha_m) \tag{38}$$

Equation (38) is the mode shape function of the system with homogeneous parameters. In this case the nodal diameters are located symmetrically. But for the modes of the system with random parameters, owing to the complexity of Eq. (37), the nodal diameters are located unsymmetrically, as shown in Fig. 5.

It can be shown that the mode shape function Eq. (37) satisfies the condition of orthogonality. Indeed, let us designate the quantities in parentheses in Eq. (37) by

$$B_{mn} = \xi b_{mn} + \eta c_{mn} + \zeta d_{mn} \tag{39}$$

then from Eqs. (26) and (30) we know

$$B_{mn} = -B_{nm} \tag{40}$$

Making use of Eqs. (19) and (40) we have

$$\int_{-\pi}^{\pi} X_m X_{m'} \mathrm{d}\theta = \int_{-\pi}^{\pi} \Big(y_m + \sum_{\substack{n=1\\ n\neq m}}^{\infty} B_{mn} y_n\Big)\Big(y_{m'} + \sum_{\substack{n'=1\\ n'\neq m'}}^{\infty} B_{m'n'} y_{n'}\Big)\mathrm{d}\theta = \int_{-\pi}^{\pi} \Big(y_m y_{m'} + \sum_{\substack{n=1\\ n\neq m}}^{\infty} B_{mn} y_n y_{m'} + \sum_{\substack{n'=1\\ n'\neq m'}}^{\infty} B_{m'n'} y_{n'} y_m$$

$$+ \text{terms of higher order}\Big)\mathrm{d}\theta = \delta_{mm'} + B_{mm'} + B_{m'm} + \ldots = \delta_{mm'} = \begin{cases} 1 & (\text{for } m = m') \\ 0 & (\text{for } m \neq m') \end{cases} \tag{41}$$

Thus the orthogonality of normal modes of the structure with random parameters is proved.

Phase Angles of Normal Modes

For the structure with random parameters, the phase angles of normal modes in Eq. (37) are not arbitrary, but are random variables determined by random functions $P(\theta)$, $Q(\theta)$, and $M(\theta)$. They are independent of initial conditions of motion.

In fact, when we solve Eqs. (16b-d), it should be noted that the right-hand side of these differential equations should not contain terms involving spatial frequency m; otherwise the solution would contain terms increasing infinitely with time and this is unreasonable. To avoid this situation, we should put the coefficient of $\sin(m\theta + \alpha_m)$ in the right-hand side of the equations to zero. From this relation we get the phase angle

$$tg2\alpha_m = \frac{-\xi p_0^2 |f_{2m}| \sin\beta_{2m} + \eta m^2 q_0^2 |g_{2m}| \sin\gamma_{2m} + \zeta\lambda_m^2 |h_{2m}| \sin\delta_{2m}}{-\xi p_0^2 |f_{2m}| \cos\beta_{2m} + \eta m^2 q_0^2 |g_{2m}| \cos\gamma_{2m} + \zeta\lambda_m^2 |h_{2m}| \cos\delta_{2m}} \tag{42}$$

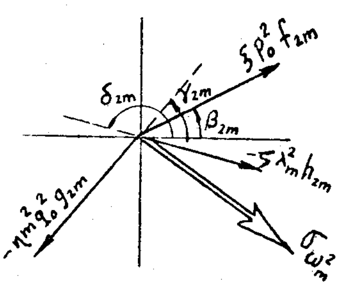

Fig. 4 Standard deviation of frequencies.

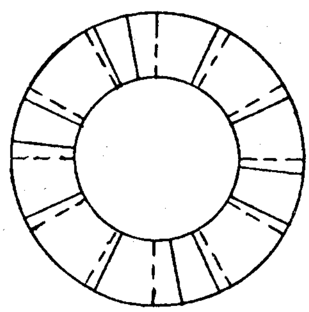

Fig. 5 Nodal diameters; —— unsymmetrical by Eq. (37); – – – unsymmetrical by Eq. (38).

Forced Vibration

When the structure described in Fig. 2 rotates with angular velocity Ω, it is acted upon by an exciting force involving many harmonics:

$$f(\theta,t) = \sum_{k=1}^{\infty} f_k(\theta,t) \tag{43}$$

To simplify the problem we examine only the response to the kth harmonic of the exciting force and assume that the spatial distribution of the exciting force is proportional to the kth normal mode $y_k(\theta)$ of the structure; thus we may write:

$$f_k(\theta,t) = m_{x_0} F_k y_k(\theta) e^{ik\Omega t} \tag{44}$$

Assume that the solution of Eq. (2) with the right-hand side of Eq. (44) takes the following form:

$$x(\theta,t) = X(\theta) e^{ik\Omega t} \tag{45}$$

Substituting it into Eq. (2), we get the spatial differential equation for the mode shape function $X(\theta)$ of the forced vibration, but for simplifying the problem it is assumed that

$m_x(\theta) = m_{x_0} = \text{const}$; thus we have:

$$-(q^2X')' + [p^2 - (k\Omega)^2]X + i\epsilon k\Omega X = F_k y_k(\theta) \tag{46}$$

Assume that the solution of differential equation (46) may be expressed as the sum of normal modes of the structure:

$$X(\theta) = \sum_{m=1}^{\infty} a_m X_m(\theta) \tag{47}$$

where the mth normal mode $X_m(\theta)$ is expressed by Eq. (37) as

$$X_m(\theta) = y_m(\theta) + \sum_{\substack{n=1 \\ n\neq m}}^{\infty} (\xi b_{mn} + \eta c_{mn}) y_n(\theta) \tag{48}$$

Substituting Eq. (47) into Eq. (46), multiplying it by $X_m(\theta)$, and integrating it, we have:

$$\int_{-\pi}^{\pi} \{ [p_0^2 - (k\Omega)^2] XX_m + \xi p_0^2 PXX_m - q_0^2 X'' X_m$$

$$- \eta q_0^2 (QX')' X_m + i\epsilon k\Omega XX_m \} \mathrm{d}\theta = \int_{-\pi}^{\pi} F_k y_k(\theta) X_m \mathrm{d}\theta \tag{49}$$

Making use of the condition of orthogonality Eq. (41) and neglecting terms of higher order, we get

$$[(p_0^2 + m^2 q_0^2 + \xi\mu_m^2 + \eta\nu_m^2) - (k\Omega)^2 + i\epsilon k\Omega] a_m$$

$$= F_k [\delta_{km} + (\xi b_{mk} + \eta c_{mk})_{k\neq m}] \tag{50}$$

Taking the expression of natural frequency Eq. (31) into account, we have

$$[\omega_m^2 - (k\Omega)^2 + i\epsilon k\Omega] a_m = \begin{cases} F_k & (\text{for } m=k) \\ F_k(\xi b_{mk} + \eta c_{mk}) & (\text{for } m\neq k) \end{cases} \tag{51}$$

Once coefficient a_m is obtained, the solution of Eq. (46) is obtained in the form of Eq. (47).

Amplitude of Resonance and Its Variance Estimation

In Ref. 2 the condition of resonance called the "triple-coincidence condition" for a circumferentially closed structure of blades is given in the following form:

$$k\Omega = \omega_m \quad \text{and} \quad k = m$$

or

$$\omega_m / \Omega = k = m$$

It means that resonance occurs only in the case when the diameters equals the frequency $k\Omega$ of the kth harmonic of diameter equals the frequency $k\Omega$ of the kth harmonic of exciting force and the number of nodal diameters m coincides with the number of harmonic k. The above condition of resonance is given only for the case of homogeneous structures.

For the case of structures with random parameters, when the structure is in resonance, i.e., when the frequency of exciting force $k\Omega$ coincides with the natural frequency ω_m of the structure

$$k\Omega = \omega_m \tag{52}$$

we should estimate the amplitude of resonance in two different cases (Fig. 6):

1) In the case where the triple-coincidence condition is satisfied:

$$k\Omega = \omega_m \quad \text{and} \quad k = m \tag{53}$$

from Eq. (51) we get a deterministic solution:

$$|a_m|_0 = \frac{F_k}{\epsilon k\Omega} \tag{54}$$

In this case, a violent resonance occurs in the case of small damping. This result is common both for the case of structure with homogeneous parameters and for that with random parameters.

2) In the case where the triple-coincidence condition is not satisfied, i.e., when

$$k\Omega = \omega_m \quad \text{but} \quad k \neq m \tag{55}$$

from Eq. (51) we have a random solution with small parameters ξ and η:

$$|a_m| = \frac{F_k}{\epsilon k\Omega} (\xi b_{mk} + \eta c_{mk})_{m\neq k} \tag{56}$$

The ratio of $|a_m|$ to $|a_m|_0$ is

$$\frac{|a_m|}{|a_m|_0} = (\xi b_{mk} + \eta c_{mk})_{m\neq k} \tag{57}$$

where coefficients b_{mk} and c_{mk} are shown in Eqs. (26) and (30).

In this case, a resonance occurs but it is a weak one even in the case of small damping because of the appearance of small parameters ξ and η in Eq. (56). This weak resonance occurs only in the case of structures with random parameters. It does not exist in the case of structures with homogeneous parameters. The mean value of the amplitude ratio of resonance in this case is zero:

$$E\left[\frac{|a_m|}{|a_m|_0}\right] = \frac{1}{2\pi}\frac{1}{m^2 - k^2}\int_{-\pi}^{\pi}\left\{\xi\frac{p_0^2}{q_0^2}E[P(\theta)]y_m y_k\right.$$

$$\left. + \eta E[Q(\theta)] y_m' y_k' \right\} \mathrm{d}\theta = 0 \tag{58}$$

The variance of the amplitude ratio of resonance in this case is expressed in the following equation:

$$\sigma_a^2 = \mathrm{var}\frac{|a_m|}{|a_m|_0} = E\left(\frac{|a_m|}{|a_m|_0}\right)^2 = \frac{1}{4\pi^2}\left(\frac{1}{m^2 - k^2}\right)^2$$

$$\times \left\{\xi^2 \frac{p_0^4}{q_0^4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} E[P(\theta_1)P(\theta_2)] y_m(\theta_1) y_k(\theta_1) y_m(\theta_2)\right.$$

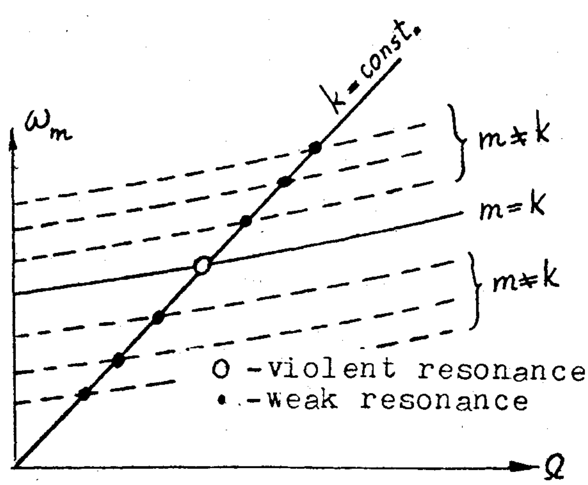

Fig. 6 Campbell diagram.

$$\times y_k(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2+\eta^2\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}E[Q(\theta_1)Q(\theta_2)]y_m'(\theta_1)$$

$$\times y_k'(\theta_1)y_m'(\theta_2)y_k'(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2$$

$$+2\xi\eta\frac{p_0^2}{q_0^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}E[P(\theta_1)Q(\theta_2)]$$

$$\left.\times y_m(\theta_1)y_k'(\theta_2)y_m'(\theta_1)y_k(\theta_2)\mathrm{d}\theta_1\mathrm{d}\theta_2\right\}\quad(m\neq k)\tag{59}$$

The result of the above integration is

$$\sigma_a^2=\mathrm{var}\ \frac{|a_m|}{|a_m|_0}=\frac{1}{2(m^2-k^2)^2}\left\{\xi^2\frac{p_0^4}{q_0^4}|f_{m+k}|^2\right.$$

$$+\eta^2m^4|g_{m+k}|^2-2\xi\eta\frac{p_0^2}{q_0^2}|f_{m+k}||g_{m+k}|\cos(\beta_{m+k}-\gamma_{m+k})$$

$$+\xi^2\frac{p_0^4}{q_0^4}|f_{m-k}|^2+\eta^2m^4|g_{m-k}|^2$$

$$\left.+2\xi\eta\frac{p_0^2}{q_0^2}|f_{m-k}||g_{m-k}|\cos(\beta_{m-k}-\gamma_{m-k})\right.\quad(m\neq k)\tag{60}$$

From the expression of the variance of the resonant amplitude it may be considered that the standard deviation is the vector sum of random Fourier coefficients, as shown in Fig. 7.

Numerical Examples

Here we give some numerical examples to illustrate how the random distribution of structural parameters affects the natural frequencies and resonant amplitudes.

Suppose the local frequency of the bars in Fig. 1 equals 110.0 Hz and the local frequency of the connecting springs equals 22.0 Hz, so we have the structural parameters as follows:

$$p_0=2\pi\cdot110=691.15\ \mathrm{rad/s}$$

$$q_0=2\pi\cdot\ 22=138.23\ \mathrm{rad/s}$$

The natural frequencies of the closed ring in Fig. 1 with m nodal diameters may be calculated by Eq. (17) and are shown in Table 1.

Suppose the structural parameters of the ring model are randomly distributed. Assume the autocorrelation function of the parameters takes the form of Eq. (10) with $\alpha=1$. So we may take the Fourier coefficients of the parameters $|f_\ell|$, $|g_\ell|$, and $|h_\ell|$ ($\ell=1,2,...$) by Eq. (11) or from Fig. 3b. In turbine manufacture, the dispersion of natural frequencies of blades on one disk is usually limited to less than $\Delta=\pm4\%$. So we

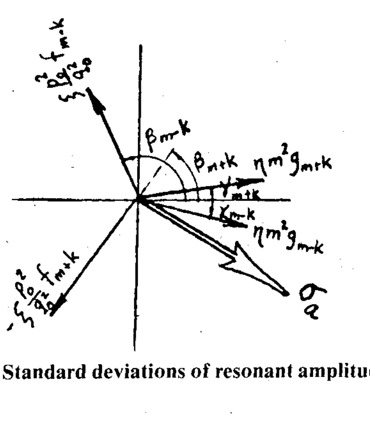

Fig. 7 Standard deviations of resonant amplitude.

Table 1 Natural frequencies

m	0	1	2	3	4	5	6
$\lambda_m(H_z)$	110.0	112.18	118.47	128.26	140.87	155.56	171.82

Table 2 Limit values of σ_a

k \ m	0	1	2	3	4
0	1	0.1413	0.0239	0.0092	0.0046
1	0.1358	1	0.0467	0.0112	0.0055
2	0.0206	0.0365	1	0.0271	0.0085
3	0.0067	0.0086	0.0231	1	0.0229
4	0.0028	0.0035	0.0060	0.0190	1
5	0.0015	0.0016	0.0026	0.0051	0.0177
6	0.0008	0.0010	0.0013	0.0023	0.0048

may take the standard deviation of local frequencies as

$$\xi=\eta=\sigma_p=\sigma_q=\Delta/3=0.01333$$

For simplicity we take $\sigma_m=\zeta=0$.

By Eq. (35) the variance (σ_ω^2) of frequencies ω_m^2 may be estimated by the following equation:

$$\sigma_\omega^2\leq½(\xi p_0^2|f_{2m}|+\eta m^2|g_{2m}|)^2$$

and the ratio of standard deviation σ_ω of frequency ω_m to the local frequency λ_m may be estimated. In this case we have

$$\frac{\sigma_\omega}{\lambda_m}\leq0.56\%\ \text{ for }\ m=1$$

$$\frac{\sigma_\omega}{\lambda_m}\leq0.41\%\ \text{ for }\ m=2$$

$$\frac{\sigma_\omega}{\lambda_m}\leq0.34\%\ \text{ for }\ m=3$$

Thus we see that the change of natural frequencies is small. By Eq. (60) the standard deviation σ_a of resonant amplitudes $|a_m|$ under the condition $m\neq k$ may be estimated. The results are shown in Table 2. It is seen that σ_a is much smaller than 1. It means that, in the case $m\neq k$, the resonance is rather weak.

Special Features of Vibrational Characteristics

The analysis of this paper shows that the vibrational characteristics of a structure with random parameters differ from that of a structure with homogeneous parameters on the following aspects:

1) The natural frequencies of a structure with homogeneous parameters are deterministic quantities, but the natural frequencies of a structure with random parameters are random variables whose mean values are equal to the natural frequencies of a structure with homogeneous parameters, and the standard deviation of frequency of mth mode depends only upon the $(2m)$th Fourier coefficients of the random parameters and may be expressed as a vector sum of their standard deviations.

2) The normal modes of a structure with homogeneous parameters have a shape of harmonic waves with symmetrically located nodal diameters, but the normal modes of a structure with random parameters have a shape involving not only a main harmonic wave, but also an infinite number of harmonics. Thus its wave shape is rather complicated, and the nodal diameters are located unsymmetrically. Despite the complication of their mode shapes, the normal modes satisfy the condition of orthogonality.

3) The phase angles of normal modes of a structure with homogeneous parameters are arbitrary. They have to be determined by initial conditions of motion. But the phase angles of a structure with random parameters are not arbitrary, but are random variables determined by random structural parameters. They are independent of initial conditions of motion.

4) For a structure with homogeneous parameters, resonance occurs only when the triple-coincidence condition described in Ref. 2 is satisfied; otherwise resonance cannot occur even in the case of coincidence of the exciting frequency with the natural frequency but $k \neq m$, because in that case no work will be done by the exciting force on the structure. But for the structure with random parameters, a strong resonance occurs when the triple-coincidence condition is satisfied, and a weak resonance occurs when the triple-coincidence condition is not satisfied but $k\Omega = \omega_m$ and $k \neq m$. The standard deviations of the resonance amplitude in that case depend upon the $(m+k)$th and $(m-k)$th Fourier coefficients of the random parameters.

Conclusion

In this paper a feasible and convenient spectral method is presented for solving the free and forced vibration problems of a periodic structure with random parameters. Expanding the spatial periodic random functions in Fourier series, it is possible to get the expressions of correlation functions, to solve the natural frequencies and normal modes, and to estimate their mean values and variances by means of these random Fourier coefficients of structural parameters. Finally, some special features of vibrational characteristics of the structure are shown.

References

[1] Hoshiya, M. and Shah, H. C., "Free Vibration of Stochastic Beam-Column," *Journal of the Eng. Mech. Div.*, ASCE, Aug. 1971, pp. 1239-1255.

[2] Huang, W., "Free and Forced Vibration of Closely Coupled Turbomachinery Blades," *AIAA Journal*, Vol. 19, July 1981, pp. 918-924.